Relations for polarization exponents (in magnetic lattice theory)

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Relations for polarization exponents

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Abstract. It is pointed out that the eight-vertex model, the triplet model and the Ashkin–Teller model have a common exponent relation $\beta_e = \frac{1}{4} - \frac{1}{4}\alpha$ that is not a consequence of scaling theory. It is proposed that this relation reflects a special symmetry of these models and this proposal is related to possible classification of two-dimensional systems.

1. Introduction

We investigate the critical behaviour of some systems in which there are two natural order parameters and suggest that the critical point in some such systems may correspond to a point of special symmetry. Since the examples that we present are mostly exact results, two-dimensional systems are considered exclusively. In particular, we consider the Ashkin–Teller model (Ashkin and Teller 1943), the eight-vertex model (Fan and Wu 1970) and the triplet model which has three-spin interactions around the faces of a triangular lattice (Wood and Griffiths 1972). There are a number of exact results known for these systems; a further account will be given in the following sections.

In general the exact results obtained have been in agreement with general scaling theories that predict relations between critical exponents. These theories arose from the work of Widom (1965), Domb and Hunter (1965) and Patashinskii and Pokrovskii (1966) that indicated a homogeneous function form of the critical equation of state. The homogeneity properties were related to length scaling by Kadanoff (1966). Following the work of Wilson (1971a, b) in using length scaling as the basis of renormalization group techniques, it has become possible to derive expansions for the values of exponents as well as for relations between them.

The importance of the symmetry of systems is emphasized by the concepts of smoothness (Griffiths 1970) and universality (Kadanoff 1971) that suggest that critical exponents should change only when there is a change in the symmetry group of the system Hamiltonian. This formulation is not sufficiently general to include the eight-vertex model which has its exponents varying continuously with interaction strengths (Baxter 1971). Kadanoff and Wegner (1971) have related this behaviour to the special scaling properties of the interaction. Suzuki (1974) has proposed the concept of weak universality to include the eight-vertex model.

The work in this paper concerns the relation

$$\beta_e = \tfrac{1}{4} - \tfrac{1}{4}\alpha \tag{1.1}$$

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where β_e is the critical exponent describing the manner in which the spontaneous polarization goes to zero at the critical point. This relation lies outside any of the scaling theories described above. It is essentially empirical, applying to the eight-vertex model (Baxter and Kelland 1974), the triplet model (Baxter *et al* 1975) and apparently to the Ashkin–Teller model (see § 2). The most plausible explanation is that it represents a symmetry of the system at the critical point. Equation (1.1) does not apply to certain less symmetric systems (Oitmaa and Enting 1975) and does not apply in three dimensions. We are not able to present a complete explanation for the relation. The present account does, however, point out a number of significant principles in the theory of critical phenomena. Equation (1.1), assuming applicability to the Ashkin–Teller model, has been used to predict the critical exponents of the four-state Potts model (Enting 1975b).

The outline of the remainder of the paper is as follows. Section 2 discusses polarization exponents and the significance of the polarization in the various systems considered. Section 3 considers (1.1) as a symmetry relation and discusses the significance of these results for the classification of systems into universality classes. The appendixes give mathematical details, appendix 1 discussing a transformation of the triplet model and appendix 2 relating a perturbation expansion for equation (1.1) in the Ashkin–Teller model to lattice–lattice scaling theory.

2. Polarization

As emphasized by Barber and Baxter (1973), the eight-vertex model has two natural order parameters, the polarization P and the magnetization M, with conjugate fields E and H. If the eight-vertex model is regarded as a system of spins $\sigma_i = \pm 1$ on a square lattice, then $M = \langle \sigma_i \rangle$ and $P = \langle \sigma_i \sigma_j \rangle$ (i, j nearest neighbours). The interactions on the eight-vertex model are two-spin interactions between second-nearest neighbours (thus giving two independent sublattices) and four-spin interactions around the faces of the lattice. This means that the polarization gives the correlation between the sublattices.

The exponents such as β, γ, δ must be subscripted to indicate whether they refer to M or P, and this is done by using the subscripts m or e (for magnetic or electric) (Enting 1973).

To describe scaling predictions we use the formalism of Hankey and Stanley (1972) and assume that the singular part of the free energy can be written as a generalized homogeneous function

$$G(\lambda^a \epsilon, \lambda^b H, \lambda^c E) = \lambda G(\epsilon, H, E), \tag{2.1}$$

whence

$$\alpha = \alpha' = (2a-1)/a \tag{2.2}$$

$$\beta_e = (1-c)/a \tag{2.3}$$

$$\gamma_e = \gamma_e' = (2c-1)/a \tag{2.4}$$

$$\delta_e = c/(1-c) \tag{2.5}$$

$$\beta_m = (1-b)/a \tag{2.6}$$

$$\gamma_m = \gamma_m' = (2b-1)/a \tag{2.7}$$

$$\delta_m = b/(1-b). \tag{2.8}$$

In addition we can make assumptions about scaling of the correlations

$$d\nu = 2-\alpha \tag{2.9}$$

$$\gamma_e = (2-\eta_e)\nu \tag{2.10}$$

$$\gamma_m = (2-\eta_m)\nu. \tag{2.11}$$

This means that all exponents can be expressed in terms of a, b, c and the dimensionality d.

For the eight-vertex model the Baxter (1971) solution for $\alpha = \alpha'$ determines a and shows that it is a continuous function of interaction strengths. The results of Barber and Baxter (1973) give β_m and indicate $b = \frac{15}{16}$. Series expansions for δ_e (Enting and Gaunt 1974) give

$$c = \tfrac{3}{4}+\tfrac{1}{4}a. \tag{2.12}$$

Combining (2.12), (2.2) and (2.3) gives equation (1.1), $\beta_e = \frac{1}{4}-\frac{1}{4}\alpha$, a relation that has been confirmed directly by Baxter and Kelland (1974). The prediction $\delta_m = 15$ obtained by substituting $b = \frac{15}{16}$ into (2.8) has been tested by Gaunt (1974). Equation (2.9) has been confirmed by Johnson *et al* (1972).

The triplet model (Wood and Griffiths 1972) consists of a triangular lattice with spins $\sigma_i = \pm 1$, and a three-spin interaction around each triangle.

The known values of the exponents are

$$\alpha = \alpha' = \tfrac{2}{3} \tag{2.13}$$

$$\nu = \tfrac{2}{3} \tag{2.14}$$

(Baxter and Wu 1973, 1974, Baxter 1974) and

$$\beta_e = \beta_m = \tfrac{1}{12} \tag{2.15}$$

(Baxter *et al* 1975). These exponents satisfy equation (1.1). The magnetization is again $\langle\sigma_i\rangle$ and the polarization is the nearest-neighbour correlation function.

While this model seems rather unlike the eight-vertex model, a simple transformation shows it to be quite similar. We sum over all configurations of spins on a third of the lattice sites to give a honeycomb lattice with two-spin and four-spin interactions. A suitable choice of interaction strengths gives a partition function that differs from the triplet model partition function only by a non-singular factor. The two-spin interaction acts between second neighbours on the honeycomb lattice, giving two independent triangular lattices coupled by the four-spin interactions. The details of the transformation are given in appendix 1.

The final example that we consider is the Ashkin–Teller (AT) model (Ashkin and Teller 1943). This is defined as having two spins σ_i, $s_i = \pm 1$ on each site. The Hamiltonian is

$$\mathscr{H}_{AT} = \sum_{\langle ij\rangle} J\sigma_i\sigma_j + J's_is_j + J_4\sigma_i\sigma_js_is_j. \tag{2.16}$$

For the general case, the work of Wu and Lin (1974) indicates that there should be two transitions. We shall consider only $J_4 \leqslant J = J'$ in which case only one transition is expected. Even for this case the critical exponents are known only for $J_4 = 0$ when the system is two independent Ising systems and for $J_4 = J = J'$ (the four-state Potts model) where the exponents have been conjectured by Enting (1975b). If we consider the Ashkin–Teller model for $J_4 \simeq 0$ we can determine the manner in which the critical exponents behave for small J_4, using the same techniques that Kadanoff and Wegner (1971) used for the eight-vertex model.

We begin by considering an arbitrary function T and assume a critical behaviour

$$T \sim t\epsilon^{y}. \tag{2.17}$$

Then

$$\frac{\mathrm{d}T}{\mathrm{d}x} = T\frac{\mathrm{d}}{\mathrm{d}x}\ln t + T\frac{\mathrm{d}y}{\mathrm{d}x}\ln \epsilon + Ty\frac{\mathrm{d}}{\mathrm{d}x}\ln \epsilon. \tag{2.18}$$

Equation (2.18) shows that the rate of change of exponent will be given by the terms in $\mathrm{d}T/\mathrm{d}x$ that behave as $T\ln\epsilon$.

For the Ashkin–Teller model we consider an operator $\hat{O}$ of the form $\hat{O}(\{\sigma_i\})\hat{O}(\{s_i\})$, and take

$$\begin{aligned}\frac{\partial}{\partial\beta J_4}\langle\hat{O}\rangle\bigg|_{J_4=0} &= \sum_{\langle ij\rangle}\langle\hat{O}(\{\sigma\})\sigma_i\sigma_j\rangle\langle\hat{O}(\{s\})s_is_j\rangle - \langle\hat{O}(\{\sigma\})\rangle\langle\hat{O}(\{s\})\rangle\langle s_is_j\rangle\langle\sigma_i\sigma_j\rangle \\ &= \sum_{\langle ij\rangle}(\langle\hat{O}(\{\sigma\})\sigma_i\sigma_j\rangle - \langle\hat{O}(\{\sigma\})\rangle\langle\sigma_i\sigma_j\rangle)(\langle\hat{O}(\{s\})s_is_j\rangle - \langle\hat{O}(\{s\})\rangle\langle s_is_j\rangle) \\ &\quad + (\langle\hat{O}(\{\sigma\})\sigma_i\sigma_j\rangle - \langle\hat{O}(\{\sigma\})\rangle\langle\sigma_i\sigma_j\rangle)\langle\hat{O}(\{s\})\rangle\langle s_is_j\rangle \\ &\quad + (\langle\hat{O}(\{s\})s_is_j\rangle - \langle\hat{O}(\{s\})\rangle\langle s_is_j\rangle)\langle\hat{O}(\{\sigma\})\rangle\langle\sigma_i\sigma_j\rangle\end{aligned} \tag{2.19}$$

where $\langle\cdots\rangle$ denotes the expectation value at $J_4 = 0$ except when this is obviously inconsistent with the explicit form of the expressions as in $(\partial/\partial\beta J_4)\langle\hat{O}\rangle$.

If on the Ising sublattices, $\langle\hat{O}(\{\sigma\})\rangle$, $\langle\hat{O}(\{s\})\rangle$ vary as ϵ^{y_0} then

$$\frac{\partial}{\partial\beta J}\langle\hat{O}(\{\sigma\})\rangle = \sum_{\langle ij\rangle}(\langle\hat{O}(\{\sigma\})\sigma_i\sigma_j\rangle - \langle\hat{O}(\{\sigma\})\rangle\langle\sigma_i\sigma_j\rangle) \sim \langle\hat{O}(\{\sigma\})\rangle\epsilon^{-1}\frac{\partial\epsilon}{\partial\beta J}y_0. \tag{2.20}$$

This means that the final two terms in (2.19) give a contribution to $(\partial/\partial\beta J_4)\langle\hat{O}\rangle$ of

$$\langle\hat{O}\rangle\epsilon^{-1}y_0\frac{\partial\epsilon}{\partial\beta J}(\langle\sigma_i\sigma_j\rangle + \langle s_is_j\rangle).$$

Examination of this expression shows that there will be a term corresponding to the second term in (2.18) because of the $\epsilon\ln\epsilon$ term in $\langle\sigma_i\sigma_j\rangle$, $\langle s_is_j\rangle$ at $J_4 = 0$. This leads to $\langle\hat{O}\rangle$ having an exponent that behaves as

$$2y_0(1 - 2J_4A/qJ^2\beta + \dots)$$

where A is the specific heat amplitude of the Ising subsystem and q is the lattice coordination number.

Essentially equivalent arguments show that functions of the form $\hat{O}(\{\sigma\}) + \hat{O}(\{s\})$ have exponents varying as

$$y_0(1 - 2J_4A/qJ^2\beta + \dots).$$

Putting

$$2J_4A/qJ^2\beta = g \tag{2.21}$$

we have

$$\beta_m = \tfrac{1}{8}(1-g+\ldots) \tag{2.22}$$

and since susceptibility is a mixture of sum and product forms of $\hat{O}$,

$$\gamma_m = \tfrac{7}{4}(1-g+\ldots), \tag{2.23}$$

whence by scaling

$$\alpha = 2g+\ldots. \tag{2.24}$$

We cannot investigate γ_e directly because $\chi_e = \partial P/\partial E$ corresponds to neither the sum form of $\langle\hat{O}\rangle$ nor the product form. The polarization is a simple product, $\sigma_i s_i$, but it is an exception to the discussion above. Not only do the final terms in (2.19) have a $\ln\epsilon$ contribution but the first term also behaves as $\ln\epsilon$, since the energy–magnetization correlation behaves as $r^{-d/2}$. This implies

$$\beta_e = \tfrac{1}{4}(1-g-h+\ldots). \tag{2.25}$$

For the square lattice the leading terms in J_4/J are the same for the Ashkin–Teller and eight-vertex models and so we have $g = h$ by using either the exact result of Baxter and Kelland (1974) or, as described by Enting and Gaunt (1974), scaling perturbation expansions based on the correlation amplitudes of Hecht (1967).

Equation (2.21) shows that g will be lattice-dependent, but if one makes an assumption of a lattice–lattice scaling of correlation amplitudes, then it can be shown that g/h should be independent of which two-dimensional lattice is considered. The detailed derivation of this result is given in appendix 2. On the basis of this lattice–lattice scaling assumption, we have for all two-dimensional lattices

$$g = h \tag{2.26}$$

and so to leading order in J_4/J,

$$\beta_e = \tfrac{1}{4}-\tfrac{1}{4}\alpha. \tag{2.27}$$

This relation apparently holds to first order in J_4/J for the Ashkin–Teller model on all two-dimensional lattices, even though the individual exponents are lattice-independent only to zeroth order. The predictions for the four-state Potts model exponents (Enting 1975b) were based on the assumption that (2.27) holds for all J_4, at least up to $J_4 = J$. Those results indicate that $J_4 = J$ is, like $J_4 = 0$, a special point at which the critical exponents are lattice-independent.

3. Symmetry and universality classes

Since the relation $\beta_e = \frac{1}{4}-\frac{1}{4}\alpha$ does not appear to be a scaling relation of any sort, the most plausible explanation is that it represents a special symmetry of the critical point. This symmetry, with P^4 behaving like the energy, is also reflected in the critical correlations, since scaling predicts that the polarization–polarization correlation exponent is $\eta_e = (1-\alpha)/(2-\alpha)$, while the energy–energy correlation exponent is $\eta_E = 4(1-\alpha)/(2-\alpha)$ (η_E is the n_E of Gunton and Buckingham 1968). If there is some such energy polarization symmetry, it would seem to hold only at the critical point and could possibly be interpreted as a symmetry of the fixed point of the renormalization group, but while so little is known about the renormalization transformation of these models, this interpretation must remain speculative.

The existence of this special symmetry complicates the question of how to classify these models. The most successful classification of cooperative systems has been the construction of universality classes according to spin dimensionality D and lattice dimensionality d. Suzuki (1974) has proposed and extended the concept to weak universality classes characterized by having the same values for exponent combinations such as β_m/ν, δ_m, $(2-\alpha)/\nu$. These classes have not yet been related to the system symmetry so that, for example, it remains unclear whether the random cluster models (Potts models), whose exponents appear to vary continuously, should be in the same weak universality class as the eight-vertex model. The difficulty would seem to be an inadequate understanding of the symmetry, since the eight-vertex model symmetry is unclear and the random cluster model formulation (Fortuin and Kasteleyn 1972) conceals the symmetry of the Potts model.

The other possibility is that the special symmetry represented by (1.1) is connected with the special properties that enable various exact solutions to be found for these models. The special nature of the critical point is brought out strikingly by the Ashkin–Teller models for which exact results can be obtained only at the critical point (Wegner 1972, Enting 1975a). This uncertainty about the validity of extrapolating from the special cases is a problem common to all such work involving specialized exact results.

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Appendix 1. Transformation of the triplet model

In transforming the triplet model we consider transformation of the high-temperature expansion. We divide the triangular lattice into three sublattices A, B, C so that each face has each vertex on a different sublattice. We want an effective Hamiltonian, $\mathscr{H}_{\text{eff}}$, such that

$$Z(\beta J_3) = T_{r_\text{A}} T_{r_\text{B}} T_{r_\text{C}} \exp(-\beta \mathscr{H}_{\text{trip}}) = T_{r_\text{A}} T_{r_\text{B}} \exp(-\beta \mathscr{H}_{\text{eff}}) f(\beta J_3) \qquad \text{(A.1)}$$

where $f(\beta J_3)$ is a non-singular function.

We consider a single C site r_g with a spin variable $g = \pm 1$ and its neighbour spins a, c, e on sublattice A and b, d, f on sublattice B.

The high-temperature expansion for Z has the form

$$(1+wabg)(1+wbcg)(1+wcdg)(1+wdeg)(1+wefg)(1+wfag) \times \text{terms independent of } g.$$

Summing over all configurations of g gives

$$[1+w^6+(w^2+w^4)(S+T+ST)]f(\beta J_3) \qquad \text{(A.2)}$$

where

$$S = ac+ce+ae \qquad \text{(A.3)}$$

$$T = bd + df + fb \tag{A.4}$$

$$w = \tanh \beta J_3. \tag{A.5}$$

The effective Hamiltonian can be chosen so that the contribution for sites a, b, c, d, e, f is

$$\mathscr{H}_{\text{eff}} = J(1+S)(1+T). \tag{A.6}$$

The corresponding terms in the high-temperature expansion are

$$\begin{aligned} f_2(\beta J)[1 &+ 35v^3 + 105v^4 + 168v^5 + 280v^6 + 435v^7 + 435v^8 + 280v^9 + 168v^{10} + 105v^{11} \\ &+ 35v^{12} + v^{15} + (S + T + ST)(v + 7v^2 + 28v^3 + 84v^4 + 189v^5 + 315v^6 \\ &+ 400v^7 + 400v^8 + 315v^9 + 189v^{10} + 84v^{11} + 28v^{12} + 7v^{13} + v^{14})] \end{aligned} \tag{A.7}$$

where

$$v = \tanh \beta J. \tag{A.8}$$

Equating the ratios of constant term to $(S+T+ST)$ term in (A.7) and (A.2) gives the relation connecting v to w.

This transformation is of additional interest because one can transform spins on the A and B triangular sublattices using the techniques described by Niemeyer and van Leeuwen (1974). The transformed lattices form a new hexagonal lattice and so one has a renormalization group transformation group that preserves the symmetry. To first order in perturbation the only non-trivial fixed point is for zero four-spin interaction and so is equivalent to the lowest-order approximation described by Niemeyer and van Leeuwen. It would be of considerable interest to see if improved approximations were able to describe the behaviour of the triplet model. When attempting to construct approximate realizations of renormalization group transformations, removing the C sublattice leads to a considerable simplification. Applying the Niemeyer and van Leeuwen transformation to each of the A, B and C sublattices introduces comparatively long-range interactions even in low-order approximations.

Additional note

The transformation removing C sublattice sites is based on the work of Baxter and Wu (1974). The interpretation in terms of an Ising model with four-spin interactions has been considered independently by a number of other workers.

Appendix 2. Lattice–lattice scaling in perturbation expansions for the Ashkin–Teller model

The function of primary concern in this derivation is $C(r, \epsilon)$ where

$$C(r, \epsilon) = \langle \sigma(r_0)\sigma(r_0 + a)\sigma(r_0 + r)\rangle - \langle \sigma(r_0)\sigma(r_0 + a)\rangle \langle \sigma(r_0 + r)\rangle. \tag{A.9}$$

We are interested in comparing h and g defined by

$$g = 2J_4 A/qJ^2\beta \tag{A.10}$$

$$\tfrac{1}{4}h\langle\sigma\rangle^2 \ln \epsilon = \beta J_4 \tfrac{1}{2}q \sum_r C(r, \epsilon)^2. \tag{A.11}$$

The factor $\frac{1}{2}q$ comes from changing a sum over bonds in (2.19) to a sum over sites in (A.11).

Following the lattice–lattice scaling theory of Betts *et al* (1971) and Ferer and Wortis (1972) we assume that for two lattices x, y

$$\lambda_x C_x(r_x, \epsilon_x) = \lambda_y C_y(r_y, \epsilon_y) \tag{A.12}$$

when

$$k_x r_x = k_y r_y \tag{A.13}$$

and

$$g_x \epsilon_x = g_y \epsilon_y \tag{A.14}$$

where g_x and n_x are as defined by Betts *et al*, and k_x is the inverse length used by Ferer and Wortis.

We also use the scaling form of the magnetization

$$m_x(\epsilon_x) = m_y(\epsilon_y) \tag{A.15}$$

and its derivative with respect to ϵ,

$$g_x^{-1} m_x'(\epsilon_x) = g_y^{-1} m_y'(\epsilon_y) \qquad \text{(assuming (A.14))} \tag{A.16}$$

$$\frac{\partial m_x}{\partial \beta J} = K_x^{-1} m_x'(\epsilon_x)$$

$$= \int \rho_x \tfrac{1}{2} q_x C_x(r_x, \epsilon_x) r_x \, dr_x = \frac{q_x}{q_y} \frac{\rho_x}{\rho_y} \frac{\lambda_y}{\lambda_x} \left(\frac{k_y}{k_x}\right)^2 \int \rho_y \tfrac{1}{2} q_y C_y(r_y, \epsilon_y) r_y \, dr_y$$

$$= \frac{q_x}{q_y} \frac{\rho_x}{\rho_y} \frac{\lambda_y}{\lambda_x} \left(\frac{k_y}{k_x}\right)^2 K_y^{-1} m_y'(\epsilon_y). \tag{A.17}$$

So

$$\frac{\lambda_x}{\lambda_y} = \frac{q_x}{q_y} \frac{\rho_x}{\rho_y} \left(\frac{k_y}{k_x}\right)^2 \frac{K_x}{K_y} \frac{g_y}{g_x}. \tag{A.18}$$

We investigate the logarithmic terms in (A.11).

$$\frac{1}{4} \frac{k_x}{\beta J_4} m_x^2(\epsilon_x) \ln \epsilon_x$$

$$= \int \rho_x \tfrac{1}{2} q_x C_x^2(r_x, \epsilon_x) r_x \, dr_x = \frac{q_x}{q_y} \frac{\rho_x}{\rho_y} \left(\frac{\lambda_y}{\lambda_x}\right)^2 \left(\frac{k_y}{k_x}\right)^2 \int \rho_y \tfrac{1}{2} q_y C_y^2(r_y, \epsilon_y) r_y \, dr_y$$

$$= \frac{1}{4} \frac{q_y}{q_x} \frac{\rho_y}{\rho_x} \left(\frac{k_x}{k_y}\right)^2 \left(\frac{K_y}{K_x}\right)^2 \left(\frac{g_x}{g_y}\right)^2 \frac{h_y}{\beta J_4} m_y^2(\epsilon_y) \ln \epsilon_y. \tag{A.19}$$

Using the strong scaling relation proposed by Ferer and Wortis,

$$\frac{n_x}{n_y} = \frac{\rho_x}{\rho_y} \left(\frac{k_y}{k_x}\right)^2 \qquad \text{(in two dimensions)}, \tag{A.20}$$

and the expression given by Betts *et al* for the specific heat amplitude

$$\frac{A_x}{A_y} = \frac{n_y}{n_x} \left(\frac{g_x}{g_y}\right)^2, \tag{A.21}$$

we have

$$\frac{h_x}{h_y} = \frac{A_x / q_x K_x^2}{A_y / q_y K_y^2}. \tag{A.22}$$

A number of steps in the development above require comment. In transforming sums to integrals we use ρ to represent the density of sites as was done by Ferer and Wortis (1972). We multiply by $\frac{1}{2}q$ to represent a density of bonds. The use of (A.20) implies that we are making the standard scaling assumption of a single relevant length applicable to all correlations.

The relation (A.22) can be put into the form

$$\frac{\beta J_4 A_x}{2q_x K_x^2 h_x} = \frac{\beta J_4 A_y}{2q_y K_y^2 h_y} = 1 \qquad \text{(using square lattice case).}$$

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